Ramanujan sums for signal processing of low-frequency noise

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An aperiodic (low-frequency) spectrum may originate from the error term in the mean value of an arithmetical function such as Möbius function or Mangoldt function, which are coding sequences for prime numbers. In the discrete Fourier transform the analyzing wave is periodic and not well suited to represent the low-frequency regime. In place we introduce a different signal processing tool based on the Ramanujan sums $c_q(n)$, well adapted to the analysis of arithmetical sequences with many resonances p/q. The sums are quasiperiodic versus the time n and aperiodic versus the order q of the resonance. Different results arise from the use of this Ramanujan-Fourier transform in the context of arithmetical and experimental signals.

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I. INTRODUCTION

"In this age of computers, it is very natural to replace the continuous with the finite. One thinks nothing about replacing the real line R with a finite circle (i.e., a finite ring Z/qZ) and similarly one replaces the real Fourier transform with the fast Fourier transform" [1].

In this paper our claim is that the discrete Fourier transform (and thus the fast Fourier transform or FFT) is well suited to the analysis of periodic or quasiperiodic sequences, but fails to discover the constructive features of aperiodic sequences, such as low-frequency noise. This claim is not new and led to alternative time series analysis methods such as Poincaré maps [2] (i.e., one-dimensional return maps of the form $x_{n+1}=f(x_n)$ or more general multidimensional maps), fractal or wavelet analysis methods [3] and autoregressive moving average models [4] to mention a few. These methods appeared in diverse contexts: turbulence, financial, ecological, physiological, and astrophysical data. For stochastic sequences such as $1/f$ electronic noise only small progress was obtained thanks to these techniques [2].

Here we introduce still another approach by considering the time series as an arithmetical sequence, that is a discrete sequence $x(n)$, $n=1\cdots t$, in which generic arithmetical functions such as $\sigma(n)$ the sum of divisors of n, $\phi(q)$ the number of irreducible fractions of denominator q, the Möbius function $\mu(n)$, or the Mangoldt function $\Lambda(n)$, may be hidden.

Recently we published a number of papers that emphasize the connection between frequency measurements and arithmetic [2,5,6]. The standard heterodyne method, which compares one oscillator of frequency $f(n)$ at time n to a reference oscillator of frequency f_0, leads to an irreducible fraction p_i/q_i of index i given from continued fraction expansions of $\nu=f(n)/f_0$ and beat signals of frequencies $F(n)=f_0q_i|\nu-p_i/q_i|$. Jumps between fractions of index i, $i\pm1$, $i\pm2$,... were clearly identified as a source of white or $1/f$ frequency noise in such frequency counting measurements [5]. A phase locked loop was characterized as well, leading to a possible relationship between $1/f$ noise close to baseband and arithmetical sequences of prime number theory [6].

We introduce Ramanujan sums as a different signal processing tool for these experimental files. In contrast to the discrete Fourier transform in which the basis functions are all roots of unity (Sec. II), the Ramanujan-Fourier transform (RFT) is defined from powers over the primitive roots of unity (Sec. III). We provide a table of known RFT's and emphasize the connection between $1/f$ noise and arithmetic. In this context considered in Sec. IV, a modified Mangoldt function plays a central role, since it connects the *golden ratio* found in the slope of the FFT to the Möbius function found in the structure of the RFT. Concerning the experimental files considered in Sec. V, galactic nuclei are promising candidates for the application of the present method.

II. THE DISCRETE FOURIER TRANSFORM

The discrete Fourier transform (DFT) or its fast analog (the FFT) is a well known signal processing tool. It extends the conventional Fourier analysis to sequences with finite period q (for the FFT one takes $q=2^l$, with l an integer).

In the DFT one starts with the roots of unity of the form $\exp[2i\pi(p/q)]$, $p=1,...,q$ and the signal analysis is performed thanks to the nth power,

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TABLE I.

$x(n)$	$\hat{x}(p)$
1	$q\ \delta_0(p)$
$e_l(n)$	$q\ \delta_l(p)$
$\delta_l(n)$	$e_l(-p)$
$\frac{1}{2}(\delta_1+\delta_{-1})(n)$	$\cos(2\pi p/q)$
$L_q(-n)$	$L_q(n)\ \hat{L}_q(-1)$

$$e_p(n)=\exp\left(2i\pi\frac{p}{q}n\right). \tag{1}$$

(In the mathematical language one says that $e_p(n)$ is a character of $G=Z/qZ$; it is a group homomorphism from the additive group G into the multiplicative group of complex numbers of norm 1.) The DFT of the time series $x(n)$ is defined as

$$\hat{x}(p)=\sum_{n=1}^{q} x(n)e_p(-n), \tag{2}$$

and there are a number of relations such as the inversion formula

$$x(n)=\frac{1}{q}\sum_{p=1}^{q} \hat{x}(p)e_p(n), \tag{3}$$

the Parseval formula (conservation of energy)

$$\sum_{n=1}^{q} |x(n)|^2=\frac{1}{q}\sum_{p=1}^{q} |\hat{x}(p)|^2, \tag{4}$$

the orthogonality relations between the characters

$$\sum_{n=1}^{q} e_p(n)e_r(n)=q\,\delta_p(r)=\begin{cases} q & \text{if } p\equiv r(\mathrm{mod}q) \\ 0 & \text{otherwise,}\end{cases} \tag{5}$$

and the convolution formula

$$\widehat{x*y}=\hat{x}\hat{y}, \tag{6}$$

where $*$ means the convolution. Generic discrete Fourier transforms are given in Table I. In particular, as is well known, an oscillating signal $e_l(n)=\exp[2i\pi(l/q)n]$ of frequency l/q transforms to a line at $p=l$ in the DFT spectrum. Inversely a line at $n=l$ in the time series transforms to an oscillating signal $\exp[-2i\pi(l/q)p]$ of frequency l/q.

A Gaussian transforms to a Gaussian through the Fourier integral. Not so well known is that the role of the Gaussian is played by the Legendre symbol in the context of the DFT [1,7]. Let us define the Legendre symbol $L_q(n)=(n/q)$ for an odd prime q as follows:

$$\left(\frac{n}{q}\right)=\begin{cases} 0 & \text{if } q \text{ divides } n \\ +1 & \text{if } n \text{ is a square modulo } q \\ & (x^2\equiv n(\mathrm{mod}q) \text{ has a solution}) \\ -1 & \text{otherwise.}\end{cases} \tag{7}$$

There are a number of relations attached to the Legendre symbol,

$$\left(\frac{n}{q}\right)=n^{(q-1)/2}(\mathrm{mod}q),$$

$$\left(\frac{-1}{q}\right)=(-1)^{(q-1)/2},$$

$$\left(\frac{q}{p}\right)\left(\frac{p}{q}\right)=(-1)^{[(p-1)/2][(q-1)/2]}$$

for distinct odd primes p,q,

$$\left(\frac{2}{q}\right)=(-1)^{(q^2-1)/8}.$$

The invariant relation for the DFT on Z/qZ is

$$\hat{L}_q(-n)=gL_q(n) \quad \text{with } g=\hat{L}_q(-1). \tag{8}$$

The Fourier coefficient at position n equals the coefficient of the original sequence up to a constant factor $g=\hat{L}_q(-1)=\sum_{p=1}^{q}(p/q)\exp[2i\pi(p/q)]$ and $g^2=(-1)^{(q-1)/2}q$.

III. THE RAMANUJAN-FOURIER TRANSFORM

Ramanujan sums $c_q(n)$ are defined as the sums of the nth powers of the qth primitive roots of the unity [8,9],

$$c_q(n)=\sum_{\substack{p=1\\(p,q)=1}}^{q} \exp\left(2i\pi\frac{p}{q}n\right), \tag{9}$$

where $(p,q)=1$ means that p and q are coprimes. It may be observed that the $c_q(n)$ are the sums over the primitive characters $e_p(n)$. The sums were introduced by Ramanujan to play the role of basis functions over which typical arithmetical functions $x(n)$ may be projected,

$$x(n)=\sum_{q=1}^{\infty} x_q c_q(n). \tag{10}$$

It should be observed that the infinite expansion with $q\to\infty$ reminds of the Fourier series analysis, rather than the discrete Fourier transform that is taken with a finite q. As a typical example the function $\sigma(n)$ (the sum of divisors of n) expands with a RFT coefficient $\sigma_q=(\pi^2 n/6)(1/q^2)$, that is,

TABLE II.

$x(n)$	x_q
$\frac{\sigma(n)}{n}$	$\frac{\pi^2}{6}\frac{1}{q^2}$
$\frac{\phi(n)}{n}$	$\frac{6}{\pi^2}\frac{\mu(q)}{\phi_2(q)}$
$b(n)=\frac{\phi(n)\Lambda(n)}{n}$	$\frac{\mu(q)}{\phi(q)}$
$C(n)$	$\left(\frac{\mu(q)}{\phi(q)}\right)^2$

$$\sigma(n)=\frac{\pi^2 n}{6}\left\{1+\frac{(-1)^n}{2^2}+\frac{2\cos(2n\pi/3)}{3^2}+\frac{2\cos(n\pi/2)}{4^2}+\cdots\right\}. \tag{11}$$

For functions $x(n)$ having a mean value

$$A_v(x)=\lim_{t\to\infty}\frac{1}{t}\sum_{n=1}^{t} x(n), \tag{12}$$

one obtains the inversion formula

$$x_q=\frac{1}{\phi(q)}A_v(x(n)c_q(n)). \tag{13}$$

More general formulas have also been derived in Ref. [10]. In the rest of the paper the coefficient x_q given in Eq. (13) will be called the Ramanujan-Fourier transform. It follows from a number of important relations. There is the multiplicative property of Ramanujan sums,

$$c_{qq'}(n)=c_q(n)c_{q'}(n) \quad \text{if } (q,q')=1, \tag{14}$$

and the orthogonality property

$$\sum_{n=1}^{qq'} c_q(n)c_{q'}(n)=1 \quad \text{if } q\neq q',$$

$$\sum_{n=1}^{q} c_q^2(n)=q\phi(q) \quad \text{otherwise}, \tag{15}$$

which reminds us of Eq. (5). It is relatively easy to evaluate Ramanujan sums from basic functions of number theory. Let us denote (q,n) as the greatest common divisor of q and n. Using the unique prime number decomposition of q and n,

$$q=\prod_i q_i^{\alpha_i} \quad (q_i \text{ prime}), \tag{16}$$

$$n=\prod_k n_k^{\beta_k} \quad (n_k \text{ prime}), \tag{17}$$

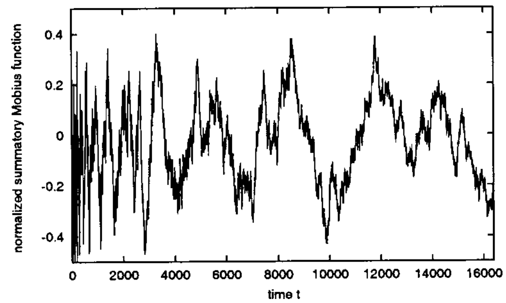

FIG. 1. The normalized summatory Möbius function $M(t)/t^{1/2}$.

one gets the number $\phi(q)$ of irreducible fractions of denominator q, also called the Euler totient function

$$\phi(q)=q\prod_i\left(1-\frac{1}{q_i}\right), \tag{18}$$

and a coding of prime numbers from the Möbius function $\mu(n)$ which is defined as

$$\mu(n)=\begin{cases}0 & \text{if } n \text{ contains a square } \beta_k>1\\ 1 & \text{if } n=1\\ (-1)^k & \text{if } n \text{ is the product}\\ & \text{of } k \text{ distinct primes.}\end{cases} \tag{19}$$

Ramanujan sums are evaluated from [11]

$$c_q(n)=\mu\left(\frac{q}{(q,n)}\right)\frac{\phi(q)}{\phi\left(\frac{q}{(q,n)}\right)}. \tag{20}$$

Note that for $(q,n)=1$, $c_q(n)=\mu(q)$. The first values are given from

$$c_1=\overline{1},\quad c_2=\overline{-1,1},\quad c_3=\overline{-1,-1,2},$$

$$c_4=\overline{0,-2,0,2},\quad c_5=\overline{-1,-1,-1,-1,4},\ldots, \tag{21}$$

where the bar indicates the period. For instance, $c_3(1)=-1$, $c_3(2)=-1$, $c_3(3)=2$, $c_3(4)=-1,\ldots$. Some generic

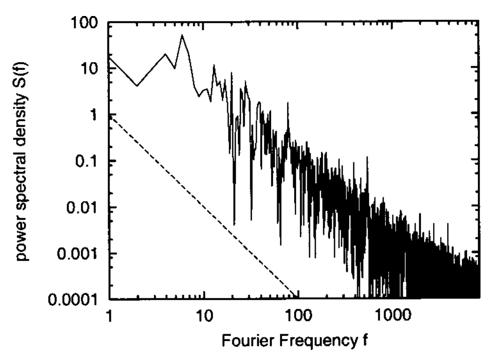

FIG. 2. The power spectral density (FFT) of the normalized summatory Möbius function $M(t)/t^{1/2}$ in comparison to the power law $1/f^2$ (dotted line).

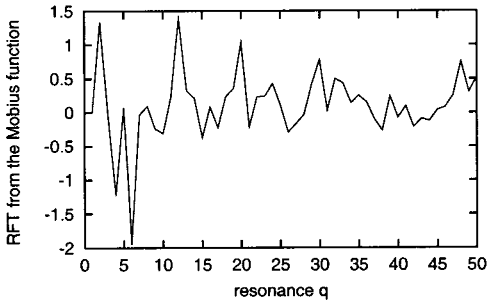

FIG. 3. The Ramanujan-Fourier transform (RFT) of the normalized summatory Möbius function shown in Fig. 1.

Ramanujan-Fourier transforms are given in Table II. In the table the function $\phi_2(q)$ generalizes the Euler function

$$\phi_2(q)=q^2\prod_i\left(1-\frac{1}{q_i^2}\right). \tag{22}$$

In $b(n)$ the Mangoldt function $\Lambda(n)$ is defined as

$$\Lambda(n)=\begin{cases}\ln p & \text{if } n=p^\alpha,\ p \text{ a prime}\\ 0 & \text{otherwise.}\end{cases} \tag{23}$$

According to Hardy and Littlewood (1922) the number of pairs of primes of the form $p,p+h$ is

$$\pi_h(x)\simeq C(h)\frac{x}{\ln^2(x)}, \tag{24}$$

with

$$C(h)=\begin{cases}2C_2\Pi_{p|h}\dfrac{p-1}{p-2} & \text{if } h \text{ odd}\\ 0 \quad \text{if } h \text{ even,}\end{cases} \tag{25}$$

where $p>2$ is a prime, and the notation $p|h$ means p divides h. The parameter $C_2\simeq 0.660...$ is the twin prime constant. It was recently conjectured [8] that this problem of prime pairs is also related to an autocorrelation function from the Wiener-Khintchine formula

$$A_v(b(n)b(n+h))=C(h). \tag{26}$$

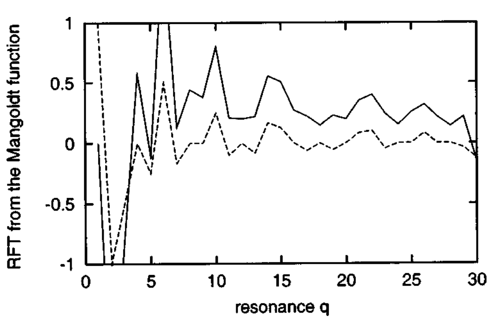

FIG. 4. Ramanujan-Fourier transform (RFT) of the error term (upper curve) of the new Mangoldt function $b(n)$ in comparison to the function $\mu(q)/\phi(q)$ (lower curve).

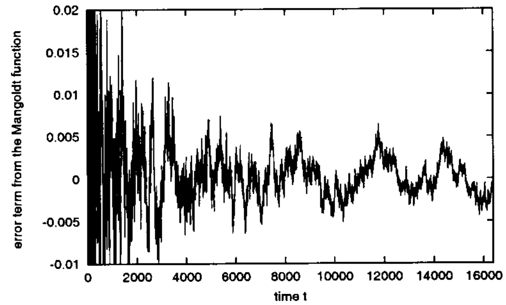

FIG. 5. Error term in the Mangoldt function $\Lambda(n)$.

IV. LOW-FREQUENCY NOISE FROM ARITHMETICAL FUNCTIONS

The idea which subtends our signal processing is that experimental signals may hide arithmetical features. It is thus very important to master the low-frequency effects due to generic arithmetical functions such as Möbius function, Mangoldt function, and so on [5,6].

A. On the summatory Möbius function

Let us consider the summatory function

$$M(t)=\sum_{n=1}^{t}\mu(n)=O(t^{1/2+\epsilon}) \quad \text{whatever } \epsilon. \tag{27}$$

The asymptotic dependence assumes the Riemann hypothesis [5]. The normalized summatory function $M(t)/t^{1/2}$ is shown in Fig. 1. The corresponding power spectral density is in Fig. 2; it looks like the FFT of a random walk since the slope is close to -2.

The RFT of $M(t)/t^{1/2}$ is shown in Fig. 3. There is no known formula for it, but it shows a signature with well defined peaks which is reminiscent of the function $\mu(q)/\phi(q)$ shown below in Fig. 4.

B. Results related to the Mangoldt function

The Riemann hypothesis can also be studied thanks to the summatory Mangoldt function,

$$\psi(t)=\sum_{n=1}^{t}\Lambda(n)=t[1+\epsilon_\psi(t)]. \tag{28}$$

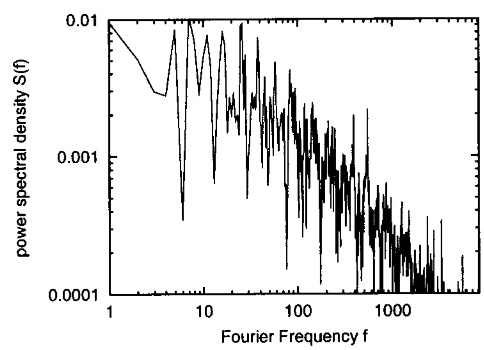

FIG. 6. Power spectral density (FFT) of the error term of Mangoldt function $\Lambda(n)$.

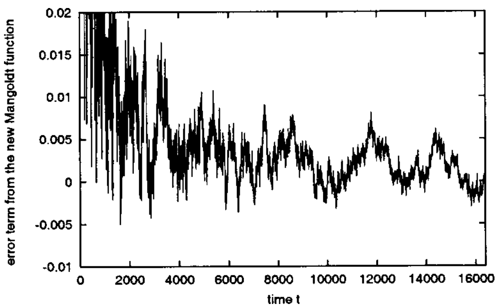

FIG. 7. Error term in the new Mangoldt function $b(n)$.

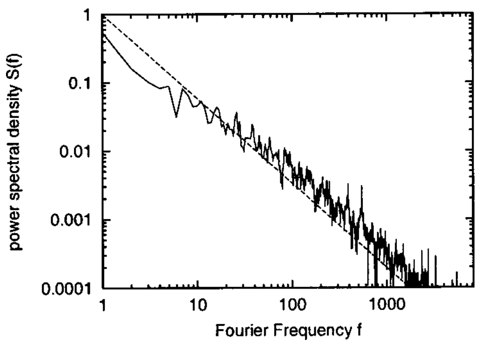

FIG. 8. Power spectral density (FFT) of the error term in the new Mangoldt function $b(n)$ in comparison to the power law $1/f^{2\alpha}$, with $\alpha = [\sqrt{(5)} - 1]/2$ the golden mean.

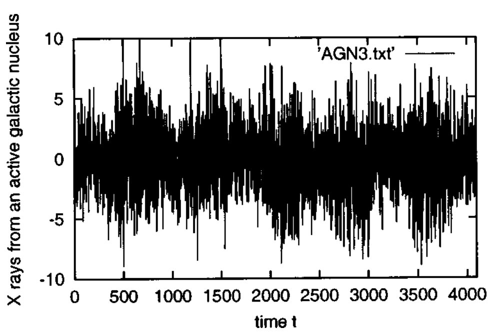

FIG. 9. X-ray variability from an active galactic nucleus (AGN). The bin time for the light curve is 30 s. The vertical axis is the normalized count rate.

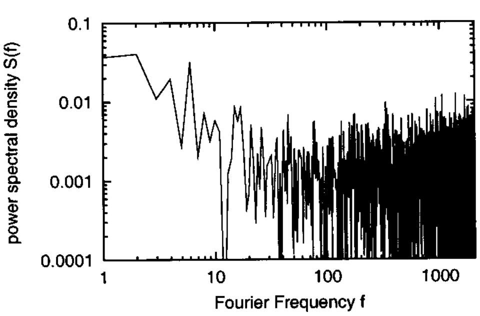

FIG. 10. Power spectral density (FFT) of the normalized x-ray variability from the AGN.

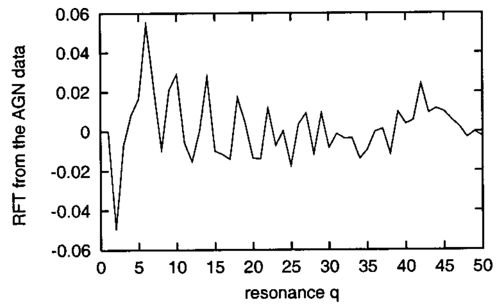

FIG. 11. Ramanujan-Fourier transform (RFT) of the normalized x-ray variability of an AGN.

The error term represented in Fig. 5 can be expressed analytically from the singularities (the pole and the zeros) of the Riemann zeta function [5]. Figure 6 shows the FFT of the error term $\epsilon_\psi(t)$: it roughly behaves as $1/f$ noise.

Hardy found that the RFT of the modified Mangoldt function $b(n) = \Lambda(n)\phi(n)/n$ equals $\mu(q)/\phi(q)$. It is thus interesting to look at the summatory function

$$B(t) = \sum_{n=1}^{t} \Lambda(n)\phi(n)/n = t[1 + \epsilon_B(t)]. \quad (29)$$

The error term in Fig. 7 is found to follow approximately the power law

$$S_B(t) \sim f^{-2\alpha} \quad (30)$$

with $\alpha = (\sqrt{5} - 1)/2 = 1/(1 + 1/[1 + 1/(1 + \cdots)])$, as shown in Fig. 8. This spectrum shows a possible connection between α and $\mu(q)$ and thus a possible relationship between the theory of diophantine approximations for quadratic irrational numbers such as α and prime number theory. The RFT of $\epsilon_B(t)$ looks similar to the one $\mu(q)/\phi(q)$ of the new Mangoldt function $b(n)$.

V. LOW FREQUENCY NOISE FROM EXPERIMENTAL DATA

Our final goal in using the Ramanujan-Fourier transform is to discover known arithmetical rules behind experimental sequences. We choose two examples.

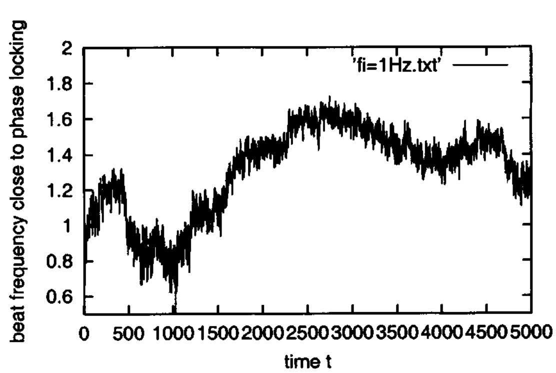

FIG. 12. Beat frequency (in Hz) between two 5 MHz radio-frequency oscillators close to phase locking versus the bin time (in s).

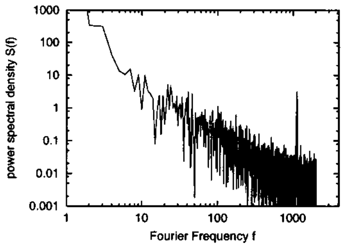

FIG. 13. FFT (in μHz) of the beat frequency for two oscillators close to phase locking versus the Fourier frequency (in Hz).

A. Low-frequency noise from galactic nuclei

The first example is taken from astronomy. The observation of variability in astronomical systems may lead to valuable information on the physical nature of the observed system. In particular Seyfert galaxies are a subset of galaxies which exhibit evidence for highly energetic phenomena in their nuclei: they are called active galactic nuclei or AGN's. They are thought to be powered by accretion onto massive black holes at their centers. X-rays are created mainly in high-temperature, high-density regimes, and since matter is fairly transparent to high-energy x-rays, monitoring x-ray emission from AGN's provides a view into the core and may be used to understand the accretion process there.

Here we used a sample of data taken from the EXOSAT archive by Koenig [12] (see Fig. 9). The power spectral density exhibits a $1/f$ low-frequency noise as well as white noise as shown in Fig. 10. The corresponding RFT analysis shown in Fig. 11 shows a well defined signature reminiscent of the RFT signature of Mangoldt function, that is, $\mu(q)/\phi(q)$. That may be an indication that many resonance processes occur between the black hole and the matter to be accreted, a process which may be described from prime number theory.

B. Low-frequency noise close to phase locking

Our second example is taken from the study of radio-frequency oscillators close to phase locking. We recently demonstrated a relation between phase locking, $1/f$ frequency noise and prime numbers [6]. According to that approach the coupling coefficient between the oscillators could be described by a Mangoldt function, leading to desynchronization effects and $1/f$ frequency noise. The RFT should be able to support that conjecture. Figures 12–14 show the beat note close to phase locking of 5-MHz oscillators, the $1/f$ noise calculated from the FFT and the corresponding RFT.

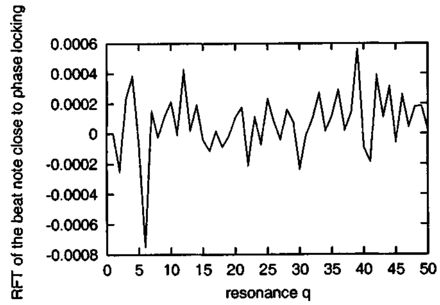

FIG. 14. RFT (in Hz) of the beat frequency for two oscillators close to phase locking.

VI. DISCUSSION

There are at least two great challenges in the study of Ramanujan sums. One can be interested in the extraction of arithmetical features from experimental files, with the aim to develop a relevant theory of their randomness. We have in mind the signals exhibiting $1/f$ noise, since this type of noise still carries much mystery in electronics as well as in other fields ranging from physics to biology and society.

The RFT signature of $1/f$ noise in a phase locked loop studied in Sec. V B still does not keep one's promise, since we were unable to relate it to a known arithmetical function. Further work is required. In contrast, high-energy astrophysics seems to be a relevant field for signal processing based on Ramanujan sums. They may help to derive plausible theories of the strong variability observed close to Seyfert or other massive galaxies. See Ref. [13] for another application of arithmetics to the black-hole remote sensing problem.

The other challenge behind Ramanujan sums relates to prime number theory. We just focused our interest to the relation between $1/f$ noise in communication circuits and the still unproved Riemann hypothesis [14]. The mean value of the modified Mangoldt function $b(n)$, introduced in Eq. (29), links Riemann zeros to the $1/f^{2\alpha}$ noise and to the Möbius function. This should follow from generic properties of the modular group SL(2,Z), the group of 2×2 matrices of determinant 1 with integer coefficients [15], and to the statistical physics of Farey spin chains [16]. See also the link to the theory of Cantorian fractal space time [17].

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[1] A. Terras, *Fourier Analysis on Finite Groups and Applications* (Cambridge University Press, Cambridge, 1999).

[2] M. Planat and C. Eckert, IEEE Trans. Ultrason. Ferroelectr. Freq. Control **47**, 1173 (2000).

[3] S. Mallat, *A Wavelet Tour of Signal Processing* (Academic, New York, 1999).

[4] R. H. Shumway and D. S. Stoffer, *Time Series Analysis and its Applications* (Springer, New York, 2000).

[5] M. Planat, Fluct. Noise Lett. **1**, R65 (2001).

[6] M. Planat and E. Henry, Appl. Phys. Lett. **80**, 2413 (2002).

[7] M. R. Schroeder, *Number Theory in Science and Communication,* Springer Series in Information Sciences (Springer, Berlin, 1999).

[8] H. G. Gadiyar and R. Padma, Physica A **269**, 503 (1999).

[9] R. M. Young, *An Introduction to Non-Harmonic Fourier Series* (Academic, New York, 2001).

[10] L. G. Lucht, Arch. Math. **64**, 121 (1995).

[11] G. H. Hardy and E. M. Wright, *An Introduction to the Theory*

of Numbers, 5th ed. (Oxford University Press, Oxford, 1979).
[12] M. Koenig, EXOSAT archive, available at http://astro.uni-tuebingen.de/groups/time/ (sample mkn766-85.dat.outZRM).
[13] H. C. Rosu and M. Planat, Mod. Phys. Lett. A **17**, 1377 (2002).
[14] M. Planat, *Noise, Oscillators and Algebraic Randomness*, Lecture Notes in Physics Vol. 550 (Springer, Berlin, 2000).
[15] S. Perrine, *La Théorie de Markoff et ses Développements* (Tessier Ashpool, Chantilly, 2002).
[16] A. Knauf, Commun. Math. Phys. **3**, 703 (1998).
[17] C. Castro and J. Mahecha, Chaos, Solitons Fractals **13**, 1407 (2002).